

## Elliptic, Parabolic and Hyperbolic Equations

Any flow configuration is the outcome of a balance between the effects of convective fluxes, diffusive fluxes and external or internal sources. The flux vector,

$$\vec{F}_D = -\kappa\rho\vec{\nabla}u \tag{1}$$

where,  $\kappa$  - is the diffusivity coefficient

From the mathematical point of view, diffusive fluxes appear through second order derivative terms in space, as a consequence of the generalized Fick law, Eqn. 1, which expresses the essence of the molecular diffusion phenomenon as a tendency to smooth out gradients. The convective fluxes, on the other hand, appear as first order derivative terms in space and express the transport properties of a flow system. Therefore each of these contributions will influence the mathematical nature of the equations, particularly the competition between the elliptic, parabolic and hyperbolic character of the systems of equations describing the approximation level being considered.

Consider one projection of the Navier-Stokes equations, say the  $x$ -component of the momentum equation, in a Cartesian system and a linear, incompressible flow under the form

$$\rho \frac{\partial u}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})u = -\frac{\partial p}{\partial x} + \mu\Delta u \tag{2}$$

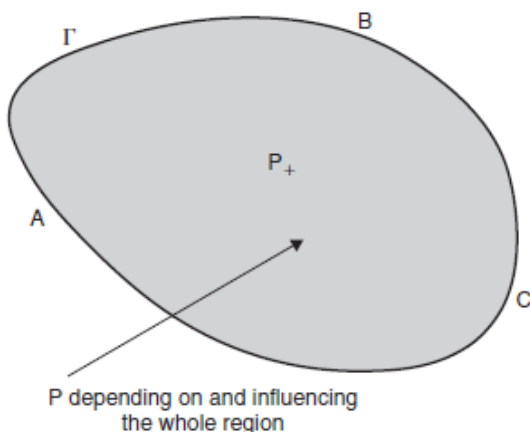
If all variables are non-dimensionalized through a reference length  $L$  for the space co-ordinate, a time scale  $T$  for the time co-ordinate, a velocity scale  $V$  for the velocity field and  $\rho V^2$  for the pressure, we obtain, keeping the same notation for all variables now considered as **non-dimensionalized**.

$$\frac{VT}{L} \frac{\partial u}{\partial t} + (\vec{v} \cdot \vec{\nabla})u = -\frac{\partial p}{\partial x} + \frac{1}{Re} \Delta u \tag{3}$$

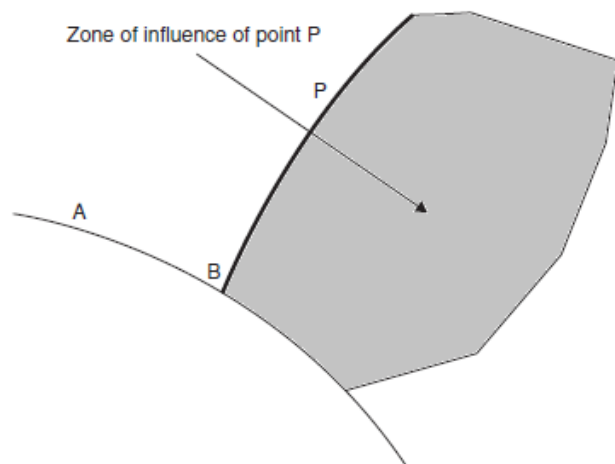
For very small Reynolds number, that is, for strongly viscous dominated flows, the convection terms can be neglected with respect to the viscous terms, and we obtain the Stokes equation

$$-\frac{V^2T}{\nu} \frac{\partial u}{\partial t} + \Delta u = Re \left( \frac{\partial p}{\partial x} \right) \tag{4}$$

This equation is purely of an elliptic type in the steady-state case for a fixed pressure gradient, but parabolic in the unsteady case due to the Laplace operator on the left-hand side. Actually, the Laplace equation (or the Poisson equation) can be considered as the standard form of an elliptic equation describing an isometric diffusion in all space directions.



Region surrounding P in an elliptic problem.



Region of dependence and zone of influence of point P for a parabolic problem with one characteristic per point.

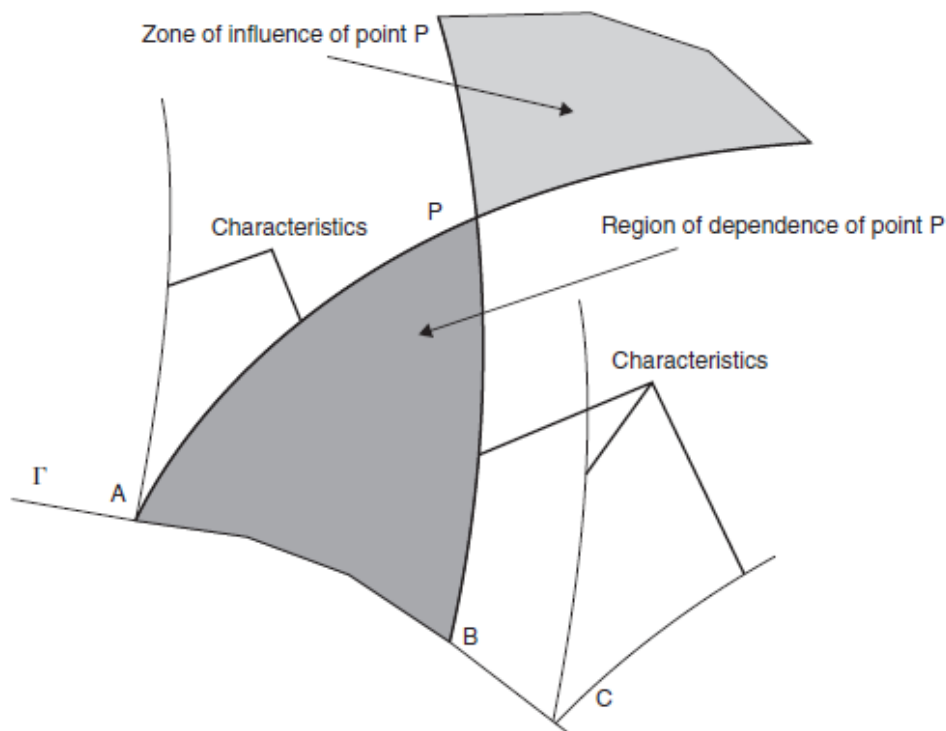
In other end, at very high Reynolds numbers and outside the boundary layers, the viscous terms have a negligible influence on the flow field, which is then dominated by the non-viscous transport terms describing the effect of the convective fluxes. Hence the equation reduces to the Euler equation

$$\frac{\partial u}{\partial t} + (\vec{v} \cdot \vec{\nabla})u = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (5)$$

which in a one-dimensional space takes the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (6)$$

and is a basic hyperbolic equation in space and time describing a propagation phenomenon.



Region of dependence and zone of influence of point P for a hyperbolic problem with two characteristics per point.

This distinction is of paramount importance, since the numerical discretization and solution methods will have to take into account the differences between phenomena as distinctive and far apart in their physical behaviour as diffusion and propagation. The former property is essentially independent of the flow direction acting in all directions and in the whole space domain, while the latter is essentially direction dominated and acts in specific regions of space defined by the wave-propagation directions. Between these two extremes the parabolic type of equations (in space and time) for a time-dependent, diffusion-dominated system

$$\left(\frac{VT}{L} Re\right) \frac{\partial u}{\partial t} = \Delta u \quad (7)$$

represents an intermediate situation between hyperbolic and elliptic. This equation, which reduces to a pure diffusive process in the steady state, describes a diffusion effect propagating in all space directions but damped in time.

Hence the system of time-dependent Navier-Stokes equation is essentially parabolic in time and space, although the continuity equation has a hyperbolic structure. Therefore they are considered as parabolic-hyperbolic. For the same reason, the steady-state form of the Navier-Stokes equations leads to elliptic-hyperbolic properties.

## The Galerkin Method

The most widely applied method is the Galerkin method in which the weighting functions are taken equal to the interpolation functions  $N_I(x)$ . This is also called the Bubnow–Galerkin method, to be distinguished from the Petrov–Galerkin method in which the test functions are different from the interpolation functions  $N_I$ .

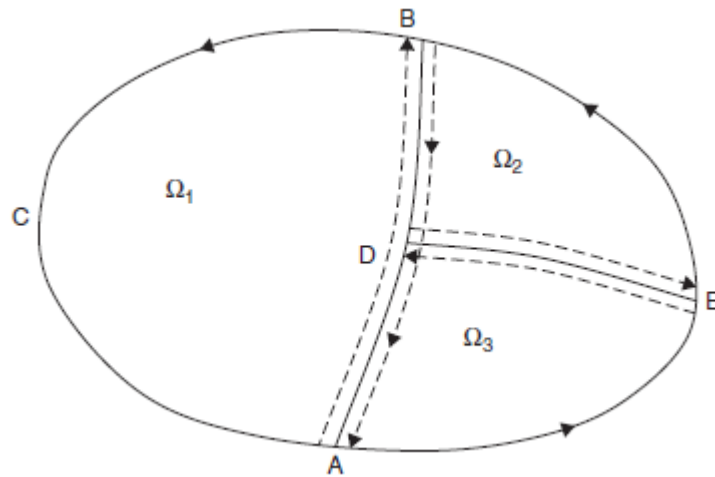
For each of the  $M$  degrees of freedom, with the finite element representation.

$$\tilde{u}(\vec{x}, t) = \sum_I u_I(t) N_I(\vec{x}) \quad (I = 1, 2, \dots, M) \quad (1)$$

and the choice  $W = N_J(x)$  in order to obtain the discretized equation for node  $J$ ,

$$- \sum_I u_I \int_{\Omega_J} [k \vec{\nabla} N_I \cdot \vec{\nabla} N_J] d\Omega + \oint_{\Gamma} k \frac{\partial \tilde{u}}{\partial n} N_J d\Gamma = \int_{\Omega_J} q N_J d\Omega \quad (2)$$

where  $\Omega_J$  is the subdomain of all elements containing node  $J$  and the summation over  $I$  covers all the nodes of  $\Omega_J$  (refer the figure).



Conservation laws for subvolumes of volume  $\Omega_1, \Omega_2, \Omega_3$ .

The matrix

$$K_{IJ} = \int_{\Omega_J} [k \vec{\nabla} N_I \cdot \vec{\nabla} N_J] d\Omega \triangleq (k \vec{\nabla} N_I, \vec{\nabla} N_J) \quad (3)$$

is called the **stiffness matrix**. For linear problems whereby  $k$  is independent of  $u$ , it will depend only on the geometry of the mesh and the chosen elements.

Eqn. 2 can also be obtained from the Rayleigh–Ritz method for homogeneous boundary conditions. This is a general property, namely, the Rayleigh–Ritz method applied to a variational formulation leads to the same system of numerical equations as the Galerkin-weighted residual method.

## The Vorticity Equation

The equations of motion can be written in many equivalent forms, one of them being obtained through the introduction of the vorticity vector  $\vec{\zeta}$ .

$$\vec{\zeta} = \vec{\nabla} \times \vec{v} \quad (1)$$

and the vector identity

$$\begin{aligned} (\vec{v} \cdot \vec{\nabla})\vec{v} &= \vec{\nabla} \left( \frac{v^2}{2} \right) - \vec{v} \times (\vec{\nabla} \times \vec{v}) \\ (\vec{v} \cdot \vec{\nabla})\vec{v} &= \vec{\nabla} \left( \frac{v^2}{2} \right) - (\vec{v} \times \vec{\zeta}) \end{aligned} \quad (2)$$

in the inertia term  $d\vec{v}/dt$  of the continuity equation. We know that the non-conservative form of the continuity equation is

$$\rho \frac{d\vec{v}}{dt} \equiv \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{\nabla} \cdot \bar{\tau} + \rho \vec{f}_e \quad (3)$$

where,  $\bar{\tau}$  - is the viscous shear stress tensor  
 $\vec{f}_e$  - is the external force vector

Substitute Eqn. 2 in Eqn. 3, and becomes

$$\begin{aligned} \rho \frac{\partial \vec{v}}{\partial t} + \rho \left[ \vec{\nabla} \left( \frac{v^2}{2} \right) - (\vec{v} \times \vec{\zeta}) \right] &= -\vec{\nabla} p + \vec{\nabla} \cdot \bar{\tau} + \rho \vec{f}_e \\ \frac{\partial \vec{v}}{\partial t} - (\vec{v} \times \vec{\zeta}) &= -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \left( \frac{v^2}{2} \right) + \frac{1}{\rho} \vec{\nabla} \cdot \bar{\tau} + \vec{f}_e \end{aligned} \quad (4)$$

This equation will be transformed further by introduction of thermodynamical relations after having discussed the conservation law for energy.

An important equation for the vorticity  $\vec{\zeta}$  can be obtained by taking the curl of the momentum equation.

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \mu \left[ \Delta \vec{v} + \frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \right] + \rho \vec{f}_e \quad (5)$$

where,  $\mu$  - is the coefficient of dynamic viscosity

This leads to the Helmholtz equation:

$$\frac{\partial \vec{\zeta}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\zeta} = \vec{\nabla} p \times \vec{\nabla} \frac{1}{\rho} + (\vec{\zeta} \cdot \vec{\nabla})\vec{v} - \vec{\zeta} (\vec{\nabla} \cdot \vec{v}) + \vec{\nabla} \times \left( \frac{1}{\rho} \vec{\nabla} \cdot \bar{\tau} \right) + \vec{\nabla} \times \vec{f}_e \quad (6)$$

For a Newtonian fluid with constant kinematic viscosity coefficient  $\nu$ , the shear stress term reduces to the Laplacian of the vorticity

$$\vec{\nabla} \times \left( \frac{1}{\rho} \vec{\nabla} \cdot \bar{\tau} \right) = \nu \Delta \vec{\zeta} \quad (7)$$

Hence, the vorticity equation is derived.